STAT 102B

TA session 5 & midterm review

Midterm will be held tomorrow, February 5, at 9:00 am in Bunche 1209 B

1. Let $\mathbf{v} = (2, 2, -1), \mathbf{w} = (1, 3, 2)$
   (a) Find the inner product of $\mathbf{v}$ and $\mathbf{w}$
   (b) Are the vectors linearly independent?
   (c) Normalize the vectors. Are they orthonormal? why?

```
v = c(2, 2, -1)
w = c(1, 3, 2)
v %*% w
vtilde = v/sqrt(sum(v^2))
wtilde = w/sqrt(sum(w^2))
vtilde %*% wtilde
```

2. Let $\mathbf{x} = (4, 1, -3), \mathbf{v_1} = (2, 0, 4), \mathbf{v_2} = (2, 0, 0)$ be three vectors in $\mathbb{R}^3$ Calculate the projection of $\mathbf{v_1}$ onto $\mathbf{v_2}$ and the corresponding residual.

```
x = c(4, 1, -3)
v1 = c(2, 0, 4)
v2 = c(2, 0, 0)
vhat1 = (v1 %*% v2)/sqrt(sum(v2^2))^2 * v2
v1 - vhat1
```

3. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} -1 & 3 & 0 \\ 5 & 1 & -2 \end{pmatrix}$$

   (a) Calculate $\mathbf{AB}, \mathbf{B}^T\mathbf{B}, \mathbf{BB}^T, det(\mathbf{A}), det(\mathbf{B}^T\mathbf{B})$
   (b) Is $\mathbf{BB}^T$ a positive definite matrix? Why or why not?

```
A = rbind(c(1, 4), c(-2, 0))
B = rbind(c(-1, 3, 0), c(5, 1, -2))
A %*% B
t(B) %*% B
B %*% t(B)
det(A)
det(t(B) %*% B)
```

4. Let X_1, X_2 be two random variables.

$$\mu_1 = E(X_1) = 6,\ \mu_2 = E(X_2) = 3$$
$$\sigma_1^2 = Var(X_1) = 3,\ \sigma_2^2 = Var(X_2) = 1$$
$$Cov(X_1, X_2) = \sigma_{1,2}^2 = \sigma_{2,1}^2 = Cov(X_2, X_1) = 2$$

Define

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

(a) Write the vector $\mu_{\mathbf{X}} = E(\mathbf{X})$ and the centered vector $\mathbf{X}_c = \mathbf{X} - \mu_{\mathbf{X}}$

(b) Write the variance-covariance matrix of $\mathbf{X}$, $\mathbf{\Sigma_X}$

(c) Find the eigenvalues of $\mathbf{\Sigma_X}$ and the corresponding eigenvectors.

(d) Let

$$\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$$

Compute the expectation and variance-covariance matrix of $\mathbf{CX}$.

```
mu = c(6, 3)
Sigma = rbind(c(3, 2), c(2, 1))
eigen(Sigma)
C = rbind(c(1, 2), c(1, 1), c(3, 2))
C %*% mu
C %*% Sigma %*% t(C)
```

5. Consider the matrix $\mathbf{X}$ and the vector y

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \; y = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 7 \end{pmatrix}$$

Find $\mathbf{X}^T\mathbf{X}$, $(\mathbf{X}^T\mathbf{X})^{-1}$, $\mathbf{X}^T y$, and $(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T y$

```
X = rbind(c(1, -1, -1), c(1, 1, -1), c(1, -1, 1), c(1, 1, 1))
y = c(4, 3, 5, 7)
t(X) %*% X
solve(t(X) %*% X)
t(X) %*% y
solve(t(X) %*% X) %*% t(X) %*% y
```

6. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & -4 \\ 2 & 1 & 4 \\ -4 & 0 & -1 \end{pmatrix}$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$

(b) Let $\mathbf{V}$ be the matrix of eigenvectors, Λ the diagonal matrix with corresponding eigenvalues and $\mathbf{V}^{-1}$ the inverse of $\mathbf{V}$. Show that $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ so $\mathbf{A}$ is diagonizable. Are the eigenvectors linearly independent?

```
A = rbind(c(-1, 0, -4), c(2, 1, 4), c(-4, 0, -1))
eigen(A)
V = eigen(A)$vectors
Lambda = diag(eigen(A)$values)
V %*% Lambda %*% t(V)
```

7. Consider the multivariate normal distribution with

$$\mu_{\mathbf{X}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{\Sigma_X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) What is the distribution of $\mathbf{Y} = \begin{pmatrix} X_1 \\ X_1 + X_2 \end{pmatrix}$?

```
mu = c(0, 0, 0)
Sigma = rbind(c(1, 0, 0), c(0, 1, 0), c(0, 0, 1))
Y = rbind(c(1, 0, 0), c(1, 1, 0))
Y %*% mu
Y %*% Sigma %*% t(Y)
```

(b) Suppose that $Z \sim N(1, 2^2)$ and is independent of all X_i. Define $Z_i = Z + X_i$ for $i = 1, 2, 3$. What is the distribution of the random vector $\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$? Determine the correlation coefficient between Z_1 and Z_2

```
mu = c(0, 0, 0, 1)
Sigma = rbind(c(1, 0, 0, 0), c(0, 1, 0, 0), c(0, 0, 1, 0), c(0, 0, 0, 4))
Y = rbind(c(1, 0, 0, 1), c(0, 1, 0, 1), c(0, 0, 1, 1))
Y %*% mu
Y %*% Sigma %*% t(Y)
```